ERGODIC THEORY AND ECONOMICS

Remarks on an article by Iván Bélyácz

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In these brief remarks, I would like to formulate a few thoughts regarding a recent study by Iván Bélyácz (Bélyácz, 2017)[1]. By reviewing elements of deterministic and stochastic dynamical systems, I will attempt on the one hand to shed clearer light on the concept of ergodicity, and on the other hand to demonstrate why there is no reason to assume that an economic system is ergodic. Finally, I will show the nature of the problem caused by a deterministic variant of rational expectations, which is avoidable in the case of naïve expectations.

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1. DETERMINISTIC DYNAMICAL SYSTEMS

Let us consider a deterministic dynamical system with discrete time (Simonovits, 1998). Let $t = 0, 1, 2, \ldots$, the index of time period, the integer $n \geq 1$ the dimension of the system, and $x_t \,\varepsilon \mathbf{R}^n$ the state of the system in the t-th period. Let us suppose that a time-variant f_t function clearly transforms the x_t state to the x_{t+1} state:

$$x_{t+1} = f_t(x_t), \qquad t = 0, 1, 2, \ldots \tag{1}$$

where the initial x_0 state is given. Thus with a simple substitution the so-called *Laplace* determinism comes about:

$$x_t = f_t(\ldots f_0(x_0)\ldots), \qquad t = 0, 1, 2, \ldots, \tag{2}$$

Put into words, if we know the initial state of a system in the t=0 initial period, then we know the state of the system in any future period.

This principle is particularly effective in natural science, where the transformation rule is generally constant in time, or in other words the system is *time-invariant*:

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots, \tag{1'}$$

so that

1 Iván Bélyácz (2017): The debated role of ergodicity in (financial) economics. *Economy & Finance*, 4 (1), pp. 1–54.

$x_t = f^t(x_0)$, $t = 0, 1, 2,...$ (2')

where f^t is the t-th iteration of the time-invariant function f.

A particularly important role in time-invariant deterministic dynamical systems is played by the *steady state* (equilibrium, fixed point), which remains in place during the transformation:

$x^o = f(x^o)$. (3)

Regular systems have only one steady state locally, but in the $x_{t+1} = x_t$ system, for example, every state is steady.

Only an *asymptotically stable* steady state has practical significance, as orbits beginning in its proximity not only remain close to the indicated state, but stay close to it asymptotically.

The simplest dynamical system is *linear*, where shifting to the appropriate system of coordinates,

$x_{t+1} = M x_t$, so that $x_t = M^t x_0$, $t = 0, 1, 2,...$ (4)

Restricting ourselves to *bounded* orbits, the absolute value of the $n \times n$ M matrix dominant eigenvalue, the spectral radius can be at most 1: $\rho(M) \leq 1$. (It should also be stipulated that in the case of equality, eigenvalues with absolute value 1 are singular.) The steady state is 0, and in the case of asymptotic stability a strict inequality prevails: $\rho(M) < 1$.

To digress: if we take into account that the initial state is only observable imprecisely in practice, then even with the simplest single-variable *non-linear* (quadratic) mapping, the system may be unpredictable. In the case of

$f(x) = 4x(1-x)$, $t = 0, 1, 2,...$ (5)

the state of the (1') system will be increasingly less predictable with the passage of time. If we look at the orbit (y_t) beginning from close to orbit (x_t), then initially the distance between the two orbits increases exponentially:

$|x_t - y_t| > \lambda^t |x_0 - y_0|$, where $\lambda > 1$.

Due to the boundedness of the orbits, the divergence ceases after a time, and then begins again. At such times we are dealing with *chaotic* dynamics.

Lagging behind the natural sciences, late by several years, chaotic dynamics penetrated economics from 1980 onwards. Chaos theory has not truly become widespread in economics, however, because random phenomena have proven more important than non-linear effects.

2. MARKOV CHAINS

The simplest non-trivial example of a stochastic process is the *finite Markov chain*, where the past only affects the future through the present (*Rényi*, 1970). For the time being we are confining ourselves to *homogenous* chains. Mathematically, we need only reinterpret the time-invariant linear system: $m_{ij} \geq 0$ is an entry of the non-negative matrix $M \geq 0$ in (4), being the probability that the system will pass in one step from the j-th state to the i-th state. Here $x_{j,t} \geq 0$ is the probability that in the t-th period, the system is in the j-th state. Expressed in a formula:

$$x_{i,t+1} = \sum_j m_{ij} x_{j,}. \qquad (4')$$

According to the theorem of total probability, the system passes from any j-th state to some other state, so that the j-th column sum is 1:

$$\sum_i m_{ij} = 1, \qquad j = 1, \ldots, n. \qquad (4'')$$

From this it follows that $\rho(M) = 1$. Moreover, the sum of state probabilities remains 1 throughout:

if $\sum_i x_{i,0} = 1$, then $\sum_i x_{i,t} = 1$.

In this way, we have limited the state space to the $n - 1$ dimensional simplex, and 0 ceases to be a steady state. It can be seen that there is at least one stationary distribution, or state vector $x^o > 0$: $x^o = M x^o$ – eigenvector. A Markov chain, however, may even have an infinite number of stationary distributions; for example, in the case of the degenerated $M = I$ (identity matrix), every distribution is stationary.

Now we arrive at the central concept of Bélyácz's study, that of ergodicity: a Markov chain is *ergodic* if for any initial state the path asymptotically converges to the stationary distribution. From this it follows that an ergodic Markov chain has only a single stationary distribution. Now let us take an example of a nonergodic Markov chain.

Let us take the following 2-dimensional M matrix: 0 on the main diagonal, 1 on the cross-diagonal.

Here the dynamics are

$$x_{1,t+1} = x_{2,t} \qquad \text{and} \qquad x_{2,t+1} = x_{1,t}\,,$$

meaning

$$x_{1,t+1} = x_{1,t-1} \qquad \text{and} \qquad x_{2,t+1} = x_{1,\,t-1}\,,$$

which is a 2-*cyclical* orbit, except if $x_{1,0} = x_{2,\,0} = 1/2$ – stationary distribution. This is the simplest example of a nonergodic system.

It is apparent that if $M > 0$, that is $m_{ij} > 0$ ($i, j = 1, ..., n$), then the Markov chain is ergodic. This proposition is easily verifiable in the 2-state case.

Ergodicity is linked to the equality of the space and time averages. In the case of a Markov chain, this means that, irrespective of the state from which we launch the system, it will have asymptotic probability x_i° at the i-th state.

Until now we have assumed that the Markov chain is homogenous, and that a given M transition matrix traces the successive distributions. But just as time-variant occurrences are common in dynamical systems, so may we also encounter time-variant transitions in Markov processes. At such times, there is generally no stationary distribution, and even less so ergodicity. So what would exclude the existence of inhomogeneous Markov chains in an economy?

3. RATIONAL EXPECTATIONS

Besides ergodicity, rational expectations are assigned an important role in modern economics. To present this there is no need for a stochastic model, as the scalar deterministic dynamics discussed in section 1 are sufficient, except that we are examining a second-order rather than a first-order system. (At such times, it is customary to speak of perfect foresight or self-fulfilling prophecies.) We limit ourselves to scalar states, $n = 1$.

3.1. With dynamic expectations

The state x_t of the t-th period, besides the state x_{t-1} of the t–1-th period, is determined by the expectations x_{t+1}^e on the state of the t+1-th period:

$$F(x_{t+1}^e, x_t, x_{t-1}) = 0, \qquad t = 0, 1, 2, ... \tag{7}$$

In the case of *rational expectations*, the forecast is precise: $x_{t+1}^e = x_{t+1}$. At this time, instead of (7),

$$F(x_{t+1}, x_t, x_{t-1}) = 0, \qquad t = 0, 1, 2, ... \tag{7R}$$

the second-order differential equation is valid. Let us suppose that the steady state is **0**:

$F(0, 0, 0) = 0.$

Hence the linearized system around the steady state is given as

$$a_1 x_{t+1} + a_2 x_t + a_3 x_{t-1} = 0, \qquad t = 0, 1, 2, ... \tag{8R}$$

where

$a_1 = F_1'(0, 0, 0) \neq 0$, $a_2 = F_2'(0, 0, 0)$ and $a_3 = F_3'(0, 0, 0)$

are in turn the corresponding partial derivatives. Rendering the system of equations (8R) explicit:

$$x_{t+1} = b_2 x_t + b_3 x_{t-1}, \qquad t = 0, 1, 2, \ldots \tag{9R}$$

emerges, where the initial values x_0 and x_{-1} are given.

Here, however, the problem of *uncertainty* arises, which *Samuelson* (1958) already observed in his overlapping generations model: x_0 is unknown. This does not trouble representatives of mainstream economics, who rid themselves of the problem with the following trick.

It is known that the solution to (9R) for a general pair of initial states is

$$x_t = \xi_1 \lambda_1^{\,t} + \xi_2 \lambda_2^{\,t}, \qquad t = 0, 1, 2, \ldots \tag{10R}$$

where λ_1 and λ_2 are roots of the following quadratic equation:

$$\lambda^2 = b_2 \lambda + b_3 = 0 \tag{11}$$

and ξ_1 and ξ_2 are coefficients to be determined.

Let us also assume that both roots are real and $|\lambda_1| < 1 < |\lambda_2|$. In this case, the right-hand side of equation (10R) would blow up except if $\xi_2 = 0$. And yet the world does not blow up, and so we accept the assumption. In the restricted equation

$$x_t = \xi_1 \lambda_1^{\,t}$$

the indeterminate ξ_1 coefficient derives from the $x_0 = \xi_1$ initial assumption.

For me this trick is unacceptable. Instead, I propose the notion of naïve expectations, somewhat discredited in recent decades, where we identify the probable future state with the present state:

$x_{t+1}^{e} = x_t$. In this case, instead of (8R), we get

$$(a_1 + a_2)\, x_t + a_3 x_{t-1} = 0, \qquad t = 0, 1, 2, \ldots. \tag{8N}$$

where the initial value x_{-1} is given. Assuming that $a_1 + a_2 \neq 0$, the uncertainty disappears.

Put into explicit form:

$$x_t = c x_{t-1}, \qquad t = 0, 1, 2, \ldots \tag{9N}$$

To summarise, ergodicity is the stochastic generalization of the stability of a time-invariant deterministic dynamical system. In the social sciences we have much less reason to assume time-invariance than in the natural sciences. At the same time, the notion of rational expectations is a very limiting assumption, even in a deterministic instance.

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